

Generalization of the Brauer Theorem to Matrix Polynomials and Matrix Laurent Series*

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Abstract

Given a square matrix A , Brauer's theorem [Duke Math. J. 19 (1952), 75–91] shows how to modify one single eigenvalue of A via a rank-one perturbation, without changing any of the remaining eigenvalues. We reformulate Brauer's theorem in functional form and provide extensions to matrix polynomials and to matrix Laurent series $A(z)$ together with generalizations to shifting a set of eigenvalues. We provide conditions under which the modified function $\tilde{A}(z)$ has a canonical factorization $\tilde{A}(z) = \tilde{U}(z)\tilde{L}(z^{-1})$ and we provide explicit expressions of the factors $\tilde{U}(z)$ and $\tilde{L}(z)$. Similar conditions and expressions are given for the factorization of $\tilde{A}(z^{-1})$. Some applications are discussed.

1 Introduction

Brauer's theorem [9] relates the eigenvalues of an $n \times n$ matrix A to the eigenvalues of the modified matrix $\tilde{A} = A + uv^*$ when either u or v coincides with a right or with a left eigenvector of A . Its original formulation can be stated in the following form.

Theorem 1.1. *Let A be an $n \times n$ complex matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Let u_k be an eigenvector of A associated with the eigenvalue λ_k , and let v be any n -dimensional vector. Then the matrix $\tilde{A} = A + u_k v^*$ has eigenvalues $\lambda_1, \dots, \lambda_{k-1}, \lambda_k + v^* u_k, \lambda_{k+1}, \dots, \lambda_n$.*

This elementary result has been applied in different contexts, more specifically to the spectral analysis of the PageRank matrix [11], to the nonnegative

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inverse eigenvalue problem [10], [21], to accelerating the convergence of iterative methods for solving matrix equations in particular quadratic equations [20], equations related to QBD Markov chains [17], [14], to M/G/1-type Markov chains [5], [3], or for algebraic Riccati equations in [15], [16], [19], and [2].

In this paper we revisit Brauer's theorem in functional form and generalize it to a non-degenerate $n \times n$ matrix function $A(z)$ analytic in the open annulus $\mathbb{A}_{r_1, r_2} = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$, where $0 \leq r_1 < r_2$, so that it can be described by a matrix Laurent series $A(z) = \sum_{i \in \mathbb{Z}} z^i A_i$ convergent for $z \in \mathbb{A}_{r_1, r_2}$. Here, non-degenerate means that $\det A(z)$ is not identically zero. Throughout the paper we assume that all the matrix functions are nondegenerate.

Let $\lambda \in \mathbb{A}_{r_1, r_2}$ be an eigenvalue of $A(z)$, i.e., such that $\det A(\lambda) = 0$ and $u \in \mathbb{C}^n$, $u \neq 0$, be a corresponding eigenvector, i.e., such that $A(\lambda)u = 0$. We prove that for any $v \in \mathbb{C}^n$ such that $v^*u = 1$ and for any $\mu \in \mathbb{C}$, the function $\tilde{A}(z) = A(z)(I + \frac{\lambda - \mu}{z - \lambda} uv^*)$ is still analytic in \mathbb{A}_{r_1, r_2} , where v^* denotes the transpose conjugate of v . Moreover, $\tilde{A}(z)$ has all the eigenvalues of $A(z)$, except for λ which, if $\mu \in \mathbb{A}_{r_1, r_2}$, is replaced by μ , otherwise is removed. We provide explicit expressions which relate the coefficients \tilde{A}_i of $\tilde{A}(z) = \sum_{i \in \mathbb{Z}} z^i \tilde{A}_i$ to those of $A(z)$.

This transformation, called *right shift*, is obtained by using a right eigenvector u of $A(z)$. A similar version, called *left shift*, is given by using a left eigenvector v of $A(z)$. Similarly, a combination of the left and the right shift, leads to a transformation called *double shift*, which enables one to shift a pair of eigenvalues by operating to the right and to the left of $A(z)$.

This result, restricted to a matrix Laurent polynomial $A(z) = \sum_{i=-h}^k z^i A_i$, provides a shifted function which is still a matrix Laurent polynomial $\tilde{A}(z) = \sum_{i=-h}^k z^i \tilde{A}_i$. In the particular case where $A(z) = zI - A$, we find that $\tilde{A}(z) = zI - \tilde{A}$ is such that \tilde{A} is the matrix given in the classical theorem of Brauer [9]. Therefore, these results provide an extension of Brauer's Theorem 1.1 to matrix Laurent series and to matrix polynomials.

A further generalization is obtained by performing the right, left, or double shift simultaneously to a packet of eigenvalues once we are given an invariant (right or left) subspace associated with this set of eigenvalues.

Assume that $A(z)$ admits a canonical factorization $A(z) = U(z)L(z^{-1})$, where $U(z) = \sum_{i=0}^{\infty} z^i U_i$, $L(z) = \sum_{i=0}^{\infty} z^i L_i$ are analytic and nonsingular for $|z| \leq 1$. An interesting issue is to find out under which assumptions the function $\tilde{A}(z)$, obtained after any kind of the above transformations, still admits a canonical factorization $\tilde{A}(z) = \tilde{U}(z)\tilde{L}(z^{-1})$. We investigate this issue and provide explicit expressions to the functions $\tilde{U}(z)$ and $\tilde{L}(z)$. We examine also the more complicated problem to determine, if they exist, the canonical factorizations of both the functions $\tilde{A}(z)$ and $\tilde{A}(z^{-1})$.

These latter results find an interesting application to determine the explicit solutions of matrix difference equations, by relying on the theory of standard triples of [13]. In fact, in the case where the matrix polynomial $A(z) = \sum_{i=0}^d z^i A_i$, which defines the matrix difference equation, has some multiple

eigenvalues, the tool of standard triples cannot be always applied. However, by shifting away the multiple eigenvalues into a set of pairwise different eigenvalues, we can transform $A(z)$ into a new matrix polynomial $\tilde{A}(z)$, which has all simple eigenvalues. This way, the matrix difference equation associated with $\tilde{A}(z)$ can be solved with the tool of standard triples and we can easily reconstruct the solution of the original matrix difference equation, associated with $A(z)$, from that associated with $\tilde{A}(z)$. This fact enable us to provide formal solutions to the Poisson problem for QBD Markov chains also in the null-recurrent case [4].

Another important issue related to the shift technique concerns solving matrix equations. In fact, for quadratic matrix polynomials, computing the canonical factorization corresponds to solving a pair of matrix equations. The same property holds, to a certain extent, also for general matrix polynomials or for certain matrix Laurent series [3]. If the matrix polynomial has a repeated eigenvalue, as it happens in null recurrent stochastic processes, then the convergence of numerical algorithms slows down and the conditioning of the sought solution deteriorates. Shifting multiple eigenvalues provides matrix functions, where the associated computational problem is much easier to solve and where the solution is better conditioned. Moreover the solution of the original equation can be obtained by means of the formulas, that we provide in this paper, relating the canonical factorizations of $A(z)$, $A(z^{-1})$ to the canonical factorizations of $\tilde{A}(z)$ and $\tilde{A}(z^{-1})$.

An interesting application concerns manipulating a matrix polynomial $A(z)$ having a singular leading coefficient. In fact, in this case $A(z)$ has some eigenvalues at infinity. By using the shift technique it is simple to construct a new matrix polynomial $\tilde{A}(z)$ having the same eigenvalues of $A(z)$ except for the eigenvalues at infinity which are replaced by some finite value, say 1. A non-singular leading coefficient allows one to apply specific algorithms that require this condition, see for instance [7], [22], [23].

The paper is organized as follows. Section 2 provides a functional formulation of Brauer's theorem, followed by its extension to matrix polynomials and to matrix Laurent series. Section 3 describes a formulation where a set of selected eigenvalues is shifted somewhere in the complex plane. Section 4 deals with the transformations that the canonical factorization of a matrix function has after that some eigenvalue has been shifted away. Section 5 deals with some specific cases and applications. In particular, we examine the possibility of shifting eigenvalues from/to infinity, we analyze right and left shifts together with their combination, that is, the double shift, and apply them to manipulating palindromic matrix polynomials. Then we analyze quadratic matrix polynomials and the associated matrix equations. In particular, we relate the solutions of the shifted equations to the ones of the original equations.

2 Generalizations and extensions

In this section, we provide some generalizations and extensions of Brauer Theorem 1.1. In particular, we give a functional formulation, from which we obtain

an extension to matrix polynomials and to matrix (Laurent) power series.

2.1 Functional formulation and extension to matrix polynomials

Let A be an $n \times n$ matrix, let λ and $u \in \mathbb{C}^n$, $u \neq 0$, be such that $Au = \lambda u$. Consider the rational function $I + \frac{\lambda - \mu}{z - \lambda}Q$, where $Q = uv^*$ and v is any nonzero vector such that $v^*u = 1$ and μ is any complex number. We have the following

Theorem 2.1. *The rational function $\tilde{A}(z) = (zI - A)(I + \frac{\lambda - \mu}{z - \lambda}Q)$ coincides with the matrix polynomial $\tilde{A}(z) = zI - \tilde{A}$, $\tilde{A} = A + (\mu - \lambda)Q$. Moreover the eigenvalues of \tilde{A} coincide with the eigenvalues of A except for λ which is replaced by μ .*

Proof. We have $\tilde{A}(z) = (zI - A)(I + \frac{\lambda - \mu}{z - \lambda}Q) = zI - A + \frac{\lambda - \mu}{z - \lambda}(zI - A)uv^* = zI - A + \frac{\lambda - \mu}{z - \lambda}(z - \lambda)uv^* = zI - (A - (\lambda - \mu)Q)$. Moreover, $\det(zI - \tilde{A}) = \det(zI - A) \det(I + \frac{\lambda - \mu}{z - \lambda}Q) = \det(zI - A)(1 + \frac{\lambda - \mu}{z - \lambda}v^*u) = \det(zI - A) \frac{z - \mu}{z - \lambda}$. This implies that the eigenvalue λ of A is replaced by μ . \square

Relying on the above functional formulation of Brauer theorem, we may prove the following extension to matrix polynomials:

Theorem 2.2. *Let $A(z) = \sum_{i=0}^d z^i A_i$ be an $n \times n$ matrix polynomial. Let $\lambda \in \mathbb{C}$ and $u \in \mathbb{C}^n$, $u \neq 0$, be such that $A(\lambda)u = 0$. For any $\mu \in \mathbb{C}$ and for any $v \in \mathbb{C}^n$ such that $v^*u = 1$, the rational function $\tilde{A}(z) = A(z)(I + \frac{\lambda - \mu}{z - \lambda}Q)$, with $Q = uv^*$, coincides with the matrix polynomial $\tilde{A}(z) = \sum_{i=0}^d z^i \tilde{A}_i$, where*

$$\tilde{A}_i = A_i + (\lambda - \mu) \sum_{k=0}^{d-i-1} \lambda^k A_{k+i+1}Q, \quad i = 0, \dots, d-1, \quad \tilde{A}_d = A_d.$$

Moreover, the eigenvalues of $\tilde{A}(z)$ coincide with those of $A(z)$ except for λ which is shifted to μ .

Proof. Since $A(\lambda)u = 0$ we find that $\tilde{A}(z) = A(z)(I + \frac{\lambda - \mu}{z - \lambda}Q) = A(z) + \frac{\lambda - \mu}{z - \lambda}(A(z) - A(\lambda))uv^*$. Moreover, since $A(z) - A(\lambda) = \sum_{i=1}^d (z^i - \lambda^i)A_i$ and $z^i - \lambda^i = (z - \lambda) \sum_{j=0}^{i-1} \lambda^{i-j-1} z^j$ we get

$$\tilde{A}(z) = A(z) + (\lambda - \mu) \sum_{i=1}^d \sum_{j=0}^{i-1} \lambda^{i-j-1} z^j A_i Q = A(z) + (\lambda - \mu) \sum_{i=0}^{d-1} z^i \sum_{k=0}^{d-i-1} \lambda^k A_{k+i+1}Q.$$

Whence we deduce that $\tilde{A}_i = A_i + (\lambda - \mu) \sum_{k=0}^{d-i-1} \lambda^k A_{k+i+1}Q$, for $i = 0, \dots, d-1$. Concerning the eigenvalues, by taking determinants in both sides of $\tilde{A}(z) = A(z)(I + \frac{\lambda - \mu}{z - \lambda}Q)$, one obtains $\det \tilde{A}(z) = \det A(z) \det(I + \frac{\lambda - \mu}{z - \lambda}v^*u) = \det A(z) \frac{z - \mu}{z - \lambda}$, which completes the proof. \square

It is interesting to observe that the function obtained by applying the shift to a matrix polynomial is still a matrix polynomial with the same degree.

A different way of proving this result for a matrix polynomial consists in applying the following three steps: 1) to extend Brauer's theorem to a linear pencil, that is, to a function of the kind $zB+C$; 2) to apply this extension to the $nd \times nd$ linear pencil $\mathcal{A}(z) = z\mathcal{B} + \mathcal{C}$ obtained by means of a Frobenius companion linearization of the matrix polynomial; in fact, it is possible to choose the vector v in such a way that the shifted pencil still keeps the Frobenius structure; 3) reformulate the problem in terms of matrix polynomial and get the extension of Brauer's theorem. In this approach we have to rely on different companion forms if we want to apply either the right or the left shift.

This approach, based on companion linearizations, does not work in the more general case of matrix Laurent series, while the functional formulation still applies as explained in the next section.

As an example of application of this extension consider the quadratic matrix polynomial $A(z) = A_0 + zA_1 + z^2A_2$ where

$$A_0 = -\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 4 & 3 \\ 1 & 4 \end{bmatrix}, \quad A_2 = -\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}.$$

Its eigenvalues are $1/3, 1/2, 1, 1$. Moreover $u = [1, 0]^*$ is an eigenvector corresponding to the eigenvalue 1. Applying the Matlab function `polyeig` to $A(z)$ provides the following approximations to the eigenvalues

```
0.333333333333333,  0.500000000000000,
0.9999999974684400,  1.000000025315599.
```

The large relative error in the approximation of the eigenvalue 1 is due to the ill conditioning of this double eigenvalue. Performing a shift of $\lambda = 1$ to $\mu = 0$, with $v = u$, so that $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, one obtains

$$\tilde{A}_0 = -\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \tilde{A}_1 = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}, \quad \tilde{A}_2 = -\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}.$$

Applying the Matlab function `polyeig` to $\tilde{A}(z)$ provides the following approximations to the eigenvalues

```
0.000000000000000,  0.333333333333333,
0.500000000000000,  1.000000000000000.
```

In this case, the approximations are accurate to the machine precision since the new polynomial has simple well conditioned eigenvalues. This transformation has required the knowledge of the double eigenvalue and of the corresponding eigenvector. For many problems, typically for matrix polynomials encountered in the analysis of stochastic processes, this information is readily available.

2.2 The case of matrix (Laurent) power series

The shift based on the functional approach, described for a matrix polynomial, can be applied to deal with nondegenerate matrix functions which are analytic on a certain region of the complex plane. Let us start with the case of a matrix power series.

Theorem 2.3. *Let $A(z) = \sum_{i=0}^{\infty} z^i A_i$ be an $n \times n$ matrix power series which is analytic in the disk $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$. Let $\lambda \in \mathbb{D}_r$ and $u \in \mathbb{C}^n$, $u \neq 0$, be such that $A(\lambda)u = 0$. For any $\mu \in \mathbb{C}$ and $v \in \mathbb{C}^n$ such that $v^*u = 1$, the function $\tilde{A}(z) = A(z)(I + \frac{\lambda-\mu}{z-\lambda}Q)$, with $Q = uv^*$, is analytic for $z \in \mathbb{D}_r$ and coincides with the matrix power series*

$$\tilde{A}(z) = \sum_{i=0}^{\infty} z^i \tilde{A}_i, \quad \tilde{A}_i = A_i + (\lambda - \mu) \sum_{k=0}^{\infty} \lambda^k A_{k+i+1}, \quad i = 0, 1, \dots$$

Moreover, $\det \tilde{A}(z) = \det A(z) \frac{z-\mu}{z-\lambda}$.

Proof. Since $A(\lambda)u = 0$, we find that $\tilde{A}(z) = A(z)(I + \frac{\lambda-\mu}{z-\lambda}Q) = A(z) + \frac{\lambda-\mu}{z-\lambda}(A(z) - A(\lambda))uv^*$. Moreover, since $A(z) - A(\lambda) = \sum_{i=1}^{\infty} (z^i - \lambda^i)A_i$ and $z^i - \lambda^i = (z - \lambda) \sum_{j=0}^{i-1} \lambda^{i-j-1} z^j$ we get

$$\tilde{A}(z) = A(z) + (\lambda - \mu) \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \lambda^{i-j-1} z^j A_i uv^*, \quad (2.1)$$

provided that the series $S(z) = \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \lambda^{i-j-1} z^j A_i$ is convergent in \mathbb{D}_r . We prove this latter property. Since $A(z)$ is convergent in \mathbb{D}_r , then for any $0 < \sigma < r$ there exists a positive matrix Γ such that $|A_i| \leq \Gamma \sigma^{-i}$ [18, Theorem 4.4c], [3, Theorem 3.6], where $|A|$ denotes the matrix whose entries are the absolute values of the entries of A and the inequality holds component-wise. Let z be such that $|\lambda| \leq |z| < r$ and let σ be such that $|z| < \sigma < r$. We have $\sum_{i=1}^{\infty} \sum_{j=0}^{i-1} |\lambda^{i-j-1} z^j| |A_i| \leq \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} |z|^{i-1} |A_i| \leq \sigma^{-1} \sum_{i=1}^{\infty} i(|z|/\sigma)^{i-1} \Gamma$, and the latter series is convergent since $0 \leq |z|/\sigma < 1$. Therefore the series $S(z)$ is absolutely convergent, and hence convergent, for $z \in \mathbb{D}_r$. Since the series is absolutely convergent, we may exchange the order of the summations so that we obtain $S(z) = \sum_{i=0}^{\infty} z^i \sum_{k=0}^{\infty} \lambda^k A_{k+i+1}$. Whence, from (2.1), we deduce that $\tilde{A}_i = A_i + (\lambda - \mu) \sum_{k=0}^{\infty} \lambda^k A_{k+i+1}$, for $i = 0, 1, \dots$. By taking determinants in both sides of $\tilde{A}(z) = A(z)(I + \frac{\lambda-\mu}{z-\lambda}Q)$, one obtains $\det \tilde{A}(z) = \det A(z) \det(I + \frac{\lambda-\mu}{z-\lambda}v^*u) = \det A(z)(1 + \frac{\lambda-\mu}{z-\mu}) = \det A(z) \frac{z-\mu}{z-\lambda}$, which completes the proof. \square

We observe that, if in the above theorem μ does not belong to \mathbb{D}_r , the eigenvalues of the function $\tilde{A}(z)$ coincide with the eigenvalues of $A(z)$, together with their multiplicities, except for λ . More specifically, if λ is a simple eigenvalue of $A(z)$, then it is not any more eigenvalue of $\tilde{A}(z)$; if λ is an eigenvalue of

multiplicity $\ell > 1$ for $A(z)$, then λ is an eigenvalue of multiplicity $\ell - 1$ for $\tilde{A}(z)$. On the other hand, if $\mu \in \mathbb{D}_r$ then in $\tilde{A}(z)$ the eigenvalue λ is replaced by μ .

A similar result can be stated for matrix Laurent series, i.e., matrix functions of the form $A(z) = \sum_{i \in \mathbb{Z}} z^i A_i$ which are analytic over $\mathbb{A}_{r_1, r_2} = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$ for $0 < r_1 < r_2$.

Theorem 2.4. *Let $A(z) = \sum_{i \in \mathbb{Z}} z^i A_i$ be an $n \times n$ matrix Laurent series which is analytic in the annulus \mathbb{A}_{r_1, r_2} . Let $\lambda \in \mathbb{A}_{r_1, r_2}$ and $u \in \mathbb{C}^n$, $u \neq 0$, be such that $A(\lambda)u = 0$. For any $v \in \mathbb{C}^n$ such that $v^*u = 1$ and for any $\mu \in \mathbb{C}$, the function*

$$\tilde{A}(z) = A(z) \left(I + \frac{\lambda - \mu}{z - \lambda} Q \right), \quad Q = uv^*, \quad (2.2)$$

is analytic for $z \in \mathbb{A}_{r_1, r_2}$ and coincides with the matrix Laurent series $\tilde{A}(z) = \sum_{i \in \mathbb{Z}} z^i \tilde{A}_i$, where

$$\begin{aligned} \tilde{A}_i &= A_i + (\lambda - \mu) \sum_{k=0}^{\infty} \lambda^k A_{k+i+1} Q, \quad \text{for } i \geq 0, \\ \tilde{A}_i &= A_i - (\lambda - \mu) \sum_{k=0}^{\infty} \lambda^{-k-1} A_{-k+i} Q, \quad \text{for } i < 0. \end{aligned} \quad (2.3)$$

Moreover, $\det \tilde{A}(z) = \det A(z) \frac{z - \mu}{z - \lambda}$.

Proof. We proceed as in the proof of Theorem 2.3. Since $A(\lambda)u = 0$, we find that $\tilde{A}(z) = A(z)(I + \frac{\lambda - \mu}{z - \lambda} Q) = A(z) + \frac{\lambda - \mu}{z - \lambda} (A(z) - A(\lambda))uv^*$. Observe that $A(z) - A(\lambda) = \sum_{i=1}^{\infty} (z^i - \lambda^i) A_i + \sum_{i=1}^{\infty} (z^{-i} - \lambda^{-i}) A_{-i}$. Moreover, we have $z^i - \lambda^i = (z - \lambda) \sum_{j=0}^{i-1} \lambda^{i-j-1} z^j$ and $z^{-i} - \lambda^{-i} = -\frac{1}{z\lambda} (z - \lambda) \sum_{j=0}^{i-1} \lambda^{-(i-j-1)} z^{-j}$. Thus we get

$$\begin{aligned} \tilde{A}(z) &= A(z) + (\lambda - \mu) S_+(z) uv^* - \frac{\lambda - \mu}{z\lambda} S_-(z) uv^*, \\ S_+(z) &= \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \lambda^{i-j-1} z^j A_i, \quad S_-(z) = \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \lambda^{-(i-j-1)} z^{-j} A_{-i}, \end{aligned} \quad (2.4)$$

provided that the series $S_+(z)$ and $S_-(z)$ are convergent in \mathbb{A}_{r_1, r_2} . According to [3, Theorem 3.6] [18, Theorem 4.4c], for any $r_1 < \sigma < r_2$ there exists a positive matrix Γ such that $|A_i| \leq \Gamma \sigma^{-i}$ for $i \in \mathbb{Z}$. Thus we may proceed as in the proof of Theorem 2.3 and conclude that the series $S_+(z)$ and $S_-(z)$ are absolutely convergent, so that we may exchange the order of the summations and get

$$S_+(z) = \sum_{i=0}^{\infty} z^i \sum_{k=0}^{\infty} \lambda^k A_{k+i+1}, \quad S_-(z) = \sum_{i=0}^{\infty} z^{-i} \sum_{k=0}^{\infty} \lambda^{-k} A_{-(k+i+1)}.$$

Whence from the first equation in (2.4) we deduce that

$$\begin{aligned}\tilde{A}_i &= A_i + (\lambda - \mu) \sum_{k=0}^{\infty} \lambda^k A_{k+i+1} uv^*, \quad \text{for } i \geq 0, \\ \tilde{A}_i &= A_i - (\lambda - \mu) \sum_{k=0}^{\infty} \lambda^{-k-1} A_{-k+i} uv^*, \quad \text{for } i < 0.\end{aligned}$$

By taking determinants in both sides of $\tilde{A}(z) = A(z)(I + \frac{\lambda-\mu}{z-\lambda}Q)$, one obtains $\det \tilde{A}(z) = \det A(z) \det(I + \frac{\lambda-\mu}{z-\lambda}v^*u) = \det A(z)(1 + \frac{\lambda-\mu}{z-\mu}) = \det A(z) \frac{z-\mu}{z-\lambda}$, which completes the proof. \square

Observe that, if $\mu \notin \mathbb{A}_{r_1, r_2}$ then the eigenvalues of $\tilde{A}(z)$ coincide with those of $A(z)$ with their multiplicities except for λ . More specifically, if λ is a simple eigenvalue of $A(z)$, then it is not any more eigenvalue of $\tilde{A}(z)$; if λ is an eigenvalue of multiplicity $\ell > 1$ for $A(z)$, then λ is an eigenvalue of multiplicity $\ell - 1$ for $\tilde{A}(z)$. On the other hand, if $\mu \in \mathbb{A}_{r_1, r_2}$ then in $\tilde{A}(z)$ the eigenvalue λ is replaced by μ .

Observe also that if $A(z) = \sum_{i=-h}^k z^i A_i$ then from (2.3) it follows that $\tilde{A}(z) = \sum_{i=-h}^k z^i \tilde{A}_i$.

3 Shifting a set of eigenvalues

In this section we provide a generalization of Brauer theorem, where we can simultaneously shift a set of eigenvalues. We will treat the case of a matrix, of a matrix polynomial and of a matrix Laurent series.

Let $A \in \mathbb{C}^{n \times n}$, $U \in \mathbb{C}^{n \times m}$ and $\Lambda \in \mathbb{C}^{m \times m}$, where $m < n$, be such that U has full rank and $AU = U\Lambda$. In other words, the eigenvalues of Λ are a subset of the eigenvalues of A and the columns of U span an invariant subspace for A corresponding to the eigenvalues of Λ . Without loss of generality we may assume that the eigenvalues of Λ are $\lambda_1, \dots, \lambda_m$ where λ_i for $i = 1, \dots, m$ are the eigenvalues of A .

We have the following generalization of Brauer's theorem

Theorem 3.1. *Let A be an $n \times n$ matrix. Let $U \in \mathbb{C}^{n \times m}$ and $\Lambda \in \mathbb{C}^{m \times m}$, where $m < n$, be such that U has full rank and $AU = U\Lambda$. Let $V \in \mathbb{C}^{n \times m}$ be a full rank matrix such that $V^*U = I$. Let $S \in \mathbb{C}^{m \times m}$ any matrix. Then the rational matrix function*

$$\tilde{A}(z) = (zI - A)(I + U(zI - \Lambda)^{-1}(\Lambda - S)V^*)$$

coincides with the linear pencil $zI - \tilde{A}$ where $\tilde{A} = A - U(\Lambda - S)V^$. Moreover, $\det \tilde{A}(z) = \det(zI - S) \frac{\det A(z)}{\det(zI - \Lambda)}$, that is, the eigenvalues of \tilde{A} are given by $\mu_1, \dots, \mu_m, \lambda_{m+1}, \dots, \lambda_n$, where μ_1, \dots, μ_m are the eigenvalues of the matrix S .*

Proof. Denote $X(z) = (zI - \Lambda)^{-1}(\Lambda - S)$. Since $AU = U\Lambda$, we have $\tilde{A}(z) = zI - A + zUX(z)V^* - U\Lambda X(z)V^* = zI - A + U(zI - \Lambda)X(z)V^*$. Whence we get $\tilde{A}(z) = zI - (A - U(\Lambda - S)V^*)$. Concerning the eigenvalues, taking determinants we have $\det \tilde{A}(z) = \det A(z) \det(I + UX(z)V^*) = \det A(z) \det(I + X(z)V^*U) = \det A(z) \det(I + X(z))$. Since $I + X(z) = (zI - \Lambda)^{-1}(zI - \Lambda + \Lambda - S)$, we obtain $\det \tilde{A}(z) = \frac{\det A(z)}{\det(zI - \Lambda)} \det(zI - S)$, which completes the proof. \square

The above results immediately extends to matrix polynomials. Assume that we are given a matrix $U \in \mathbb{C}^{n \times m}$ and $\Lambda \in \mathbb{C}^{m \times m}$, where $m < n$, U has full rank and $\sum_{i=0}^d A_i U \Lambda^i = 0$. Then if w is an eigenvector of Λ corresponding to the eigenvalue λ , i.e., $\Lambda w = \lambda w$ then $0 = \sum_{i=0}^d A_i U \Lambda^i w = \sum_{i=0}^d A_i U \lambda^i w$, that is, $A(\lambda)Uw = 0$ so that λ is eigenvalue of $A(z)$ and Uw is eigenvector. We have the following result.

Theorem 3.2. *Let $A(z) = \sum_{i=0}^d z^i A_i$ be an $n \times n$ matrix polynomial. Let $U \in \mathbb{C}^{n \times m}$ and $\Lambda \in \mathbb{C}^{m \times m}$, where $m < n$, be such that U has full rank and $\sum_{i=0}^d A_i U \Lambda^i = 0$. Let $V \in \mathbb{C}^{n \times m}$ be a full rank matrix such that $V^*U = I$. Let $S \in \mathbb{C}^{m \times m}$ be any matrix. Then the rational function*

$$\tilde{A}(z) = A(z)(I + U(zI - \Lambda)^{-1}(\Lambda - S)V^*)$$

coincides with the matrix polynomial $\tilde{A}(z) = \sum_{i=0}^d z^i \tilde{A}_i$,

$$\tilde{A}_i = A_i + \sum_{k=0}^{d-j-1} A_{k+j+1} U \Lambda^k (\Lambda - S) V^*, \quad j = 0, \dots, d-1, \quad \tilde{A}_d = A_d.$$

Moreover, $\det \tilde{A}(z) = \det(zI - S) \frac{\det A(z)}{\det(zI - \Lambda)}$, that is, the eigenvalues of $\tilde{A}(z)$ coincide with those of $A(z)$ except for those coinciding with the eigenvalues of Λ which are replaced by the eigenvalues of S .

Proof. Denote $X(z) = (zI - \Lambda)^{-1}(\Lambda - S)$. Since $\sum_{i=0}^d A_i U \Lambda^i = 0$ we find that

$$\begin{aligned} \tilde{A}(z) &= A(z)(I + UX(z)V^*) = A(z) + A(z)UX(z)V^* \\ &= A(z) + (A(z)U - \sum_{i=0}^d A_i U \Lambda^i)X(z)V^*. \end{aligned}$$

Moreover, since $A(z)U - \sum_{i=0}^d A_i U \Lambda^i = \sum_{i=1}^d A_i U (z^i I - \Lambda^i)$ and $z^i I - \Lambda^i = (zI - \Lambda) \sum_{j=0}^{i-1} z^j \Lambda^{i-j-1}$ we get

$$\begin{aligned} \tilde{A}(z) &= A(z) + \left(\sum_{i=1}^d A_i U (zI - \Lambda) \sum_{j=0}^{i-1} z^j \Lambda^{i-j-1} \right) X(z)V^* \\ &= A(z) + \left(\sum_{i=1}^d A_i U \sum_{j=0}^{i-1} z^j \Lambda^{i-j-1} \right) (\Lambda - S)V^* \\ &= A(z) + \sum_{j=0}^{d-1} z^j \sum_{k=0}^{d-j-1} A_{k+j+1} U \Lambda^k (\Lambda - S)V^*. \end{aligned}$$

Whence we deduce that $\tilde{A}_i = A_i + \sum_{k=0}^{d-j-1} A_{k+j+1} U \Lambda^k (\Lambda - S) V^*$, $i = 0, \dots, d-1$, $\tilde{A}_d = A_d$. Concerning the eigenvalues, taking determinants we have $\det \tilde{A}(z) = \det A(z) \det(I + U X(z) V^*) = \det A(z) \det(I + X(z) V^* U) = \det A(z) \det(I + X(z))$. Since $I + X(z) = (zI - \Lambda)^{-1}(zI - \Lambda + \Lambda - S)$, we obtain $\det \tilde{A}(z) = \frac{\det A(z)}{\det(zI - \Lambda)} \det(zI - S)$, which completes the proof. \square

Similarly, we may prove the following extensions of the multishift to matrix power series and to matrix Laurent series. Below, we report only the case of a matrix Laurent series.

Theorem 3.3. *Let $A(z) = \sum_{i \in \mathbb{Z}} z^i A_i$ be a matrix Laurent series analytic over the annulus \mathbb{A}_{r_1, r_2} where $r_1 < r_2$. Let $U \in \mathbb{C}^{n \times m}$ and $\Lambda \in \mathbb{C}^{m \times m}$, where $m < n$, such that U has full rank and $\sum_{i \in \mathbb{Z}} A_i U \Lambda^i = 0$. Let $V \in \mathbb{C}^{n \times m}$ be a full rank matrix such that $V^* U = I$. Let $S \in \mathbb{C}^{m \times m}$ be any matrix. Then the matrix function*

$$\tilde{A}(z) = A(z)(I + U(zI - \Lambda)^{-1}(\Lambda - S)V^*)$$

coincides with the matrix Laurent series $\tilde{A}(z) = \sum_{i \in \mathbb{Z}} z^i \tilde{A}_i$,

$$\begin{aligned} \tilde{A}_i &= A_i + \sum_{k=0}^{\infty} A_{k+i+1} U \Lambda^k (\Lambda - S) V^*, \quad i \geq 0, \\ \tilde{A}_i &= A_i - \sum_{k=0}^{\infty} A_{-k+i} U \Lambda^{-k+1} (\Lambda - S) V^*, \quad i < 0. \end{aligned}$$

Moreover, $\det \tilde{A}(z) = \det(zI - S) \frac{\det A(z)}{\det(zI - \Lambda)}$.

Observe that if the matrix S in the above theorem has eigenvalues μ_1, \dots, μ_m where $\mu_1, \dots, \mu_k \in \mathbb{A}_{r_1, r_2}$ and $\mu_{k+1}, \dots, \mu_m \notin \mathbb{A}_{r_1, r_2}$, then the matrix function $\tilde{A}(z)$ is still defined and analytic in \mathbb{A}_{r_1, r_2} . The eigenvalues of $\tilde{A}(z)$ coincide with those of $A(z)$ except for $\lambda_1, \dots, \lambda_m$ whose multiplicity is decreased by 1, moreover, $\tilde{A}(z)$ has the additional eigenvalues $\mu_1, \dots, \mu_k \in \mathbb{A}_{r_1, r_2}$.

4 Brauer's theorem and canonical factorizations

Assume we are given a matrix Laurent series $A(z) = \sum_{i \in \mathbb{Z}} z^i A_i$, analytic over \mathbb{A}_{r_1, r_2} , $0 < r_1 < 1 < r_2$, which admits the canonical factorization [8]

$$A(z) = U(z)L(z^{-1}), \quad U(z) = \sum_{i=0}^{\infty} z^i U_i, \quad L(z) = \sum_{i=0}^{\infty} z^i L_i, \quad (4.1)$$

where $U(z)$ and $L(z)$ are invertible for $|z| \leq 1$. We observe that from the invertibility of $U(z)$ and $L(z)$ for $|z| \leq 1$ it follows that $\det L_0 \neq 0$, $\det U_0 \neq 0$. Therefore, the canonical factorization can be written in the form $A(z) = \tilde{U}(z)K\tilde{L}(z^{-1})$ where $K = U_0 L_0$ and $\tilde{U}(z) = U(z)U_0^{-1}$, $\tilde{L}(z) = L_0^{-1}L(z)$. The factorization in this form is unique [12].

The goal of this section is to find out if, and under which conditions, this canonical factorization is preserved under the shift transformation, and to relate the canonical factorization of $\tilde{A}(z)$ to that of $A(z)$.

This problem has been recently analyzed in the framework of stochastic processes in [4] for a quadratic matrix polynomial and used to provide an analytic solution of the Poisson problem.

We need a preliminary result whose proof is given in [4].

Theorem 4.1. *Let $A(z) = z^{-1}A_{-1} + A_0 + zA_1$ be an $n \times n$ matrix Laurent polynomial, invertible in the annulus \mathbb{A}_{r_1, r_2} having the canonical factorization*

$$A(z) = (I - zR_+)K_+(I - z^{-1}G_+).$$

We have the following properties:

1. *Let $H(z) := A(z)^{-1}$. Then, for $z \in \mathbb{A}_{r_1, r_2}$ it holds that $H(z) = \sum_{i=-\infty}^{+\infty} z^i H_i$ where*

$$H_i = \begin{cases} G_+^{-i} H_0, & \text{for } i < 0, \\ \sum_{j=0}^{+\infty} G_+^j K_+^{-1} R_+^j, & \text{for } i = 0, \\ H_0 R_+^i, & \text{for } i > 0. \end{cases}$$

2. *If H_0 is nonsingular, then $A(z^{-1})$ has the canonical factorization $A(z^{-1}) = (I - zR_-)K_-(I - z^{-1}G_-)$, where $K_- = A_0 + A_{-1}G_- = A_0 + R_-A_1$ and $G_- = H_0R_+H_0^{-1}$, $R_- = H_0^{-1}G_+H_0$.*

Now we can prove the following result concerning the canonical factorization of $\tilde{A}(z)$.

Theorem 4.2. *Assume that $A(z) = \sum_{i \in \mathbb{Z}} z^i A_i$ admits the canonical factorization (4.1). Let $\lambda \in \mathbb{A}_{r_1, r_2}$ and $u \in \mathbb{C}^n$, $u \neq 0$, be such that $A(\lambda)u = 0$ and $|\lambda| < 1$. Let $v \in \mathbb{C}^n$ be such that $v^*u = 1$ and let $\mu \in \mathbb{C}$ with $|\mu| < 1$. Then the matrix Laurent polynomial $\tilde{A}(z) = A(z)(I + \frac{\lambda - \mu}{z - \lambda}Q)$, where $Q = uv^*$, admits the following canonical factorization*

$$\tilde{A}(z) = \tilde{U}(z)\tilde{L}(z^{-1})$$

where $\tilde{U}(z) = U(z)$, $\tilde{L}(z) = \sum_{i=0}^{\infty} z^i \tilde{L}_i$, with

$$\tilde{L}_0 = L_0, \quad \tilde{L}_i = L_i - (\lambda - \mu) \sum_{j=1}^{\infty} \lambda^{-j} L_{j+i-1} Q, \quad i \geq 1.$$

Moreover, $\det \tilde{L}(z^{-1}) = \det L(z^{-1}) \frac{z - \mu}{z - \lambda}$.

Proof. Observe that the condition $A(\lambda)u = 0$ implies that $U(\lambda)L(\lambda^{-1})u = 0$. Since $|\lambda| < 1$ then $U(\lambda)$ is nonsingular, therefore $L(\lambda^{-1})u = 0$. Define $\tilde{L}(z^{-1}) = L(z^{-1})(I + \frac{\lambda - \mu}{z - \lambda}Q)$. In view of Theorem 2.4, the matrix function $\tilde{L}(z^{-1})$ is a matrix power series in z^{-1} , obtained by applying the right shift to $L(z^{-1})$. Moreover, the expression for its matrix coefficients follows from (2.3). By taking

the determinant of $\tilde{L}(z)$ one obtains $\det \tilde{L}(z^{-1}) = \det L(z^{-1}) \det(I + \frac{\lambda - \mu}{z - \lambda} Q) = \det L(z^{-1}) \frac{z - \mu}{z - \lambda}$. Therefore, since $\det \tilde{L}(z) = \det L(z) \frac{z^{-1} - \mu}{z^{-1} - \lambda}$, the function $\tilde{L}(z)$ is nonsingular for $|z| \leq 1$, so that $\tilde{A}(z) = \tilde{U}(z) \tilde{L}(z^{-1})$ is a canonical factorization. \square

The above theorem relates the canonical factorization of $A(z)$ with the canonical factorization of the shifted function $\tilde{A}(z)$. Assume we are given a canonical factorization of $A(z^{-1})$, it is a natural question to figure out if also $\tilde{A}(z^{-1})$ admits a canonical factorization, and if it is related to the one of $A(z^{-1})$. This issue is motivated by applications in the solution of the Poisson problem in stochastic models, where both the canonical factorizations of $\tilde{A}(z)$ and $\tilde{A}(z^{-1})$ are needed for the existence of the solution and for providing its explicit expression [1].

We give answer to this question in the case of matrix Laurent polynomials of the kind $A(z) = z^{-1}A_{-1} + A_0 + zA_1$.

Theorem 4.3. *Let $A(z) = z^{-1}A_{-1} + A_0 + zA_1$ be analytic in the annulus \mathbb{A}_{r_1, r_2} . Assume that $A(z)$ and $A(z^{-1})$ admit the canonical factorizations*

$$\begin{aligned} A(z) &= (I - zR_+)K_+(I - z^{-1}G_+), \\ A(z^{-1}) &= (I - zR_-)K_-(I - z^{-1}G_-). \end{aligned}$$

Let $\lambda \in \mathbb{A}_{r_1, r_2}$, $|\lambda| < 1$, and $u \in \mathbb{C}^n$, $u \neq 0$, be such that $A(\lambda)u = 0$. Let $\mu \in \mathbb{C}$, $|\mu| < 1$, and v be any vector such that: $v^*u = 1$, $(\lambda - \mu)v^*G_-u \neq 1$. Set $Q = uv^*$, and define the matrix Laurent polynomial $\tilde{A}(z) = z^{-1}\tilde{A}_{-1} + \tilde{A}_0 + z\tilde{A}_1$ as in (2.2). Then $\tilde{A}_{-1} = A_{-1} + (\lambda - \mu)(A_0 + \lambda A_1)Q$, $\tilde{A}_0 = A_0 + (\lambda - \mu)A_1Q$, $\tilde{A}_1 = A_1$, moreover $\tilde{A}(z)$ and $\tilde{A}(z^{-1})$ have the canonical factorizations

$$\begin{aligned} \tilde{A}(z) &= (I - z\tilde{R}_+)\tilde{K}_+(I - z^{-1}\tilde{G}_+), \\ \tilde{A}(z^{-1}) &= (I - z\tilde{R}_-)\tilde{K}_-(I - z^{-1}\tilde{G}_-), \end{aligned} \tag{4.2}$$

where $\tilde{R}_+ = R_+$, $\tilde{K}_+ = K_+$, $\tilde{G}_+ = G_+ + (\mu - \lambda)Q$, $\tilde{R}_- = \tilde{W}^{-1}\tilde{G}_+\tilde{W}$, $\tilde{G}_- = \tilde{W}R_+\tilde{W}^{-1}$, $\tilde{K}_- = \tilde{A}_0 + \tilde{A}_{-1}\tilde{G}_- = \tilde{A}_0 + \tilde{R}_-A_1$, with $\tilde{W} = W + (\mu - \lambda)QWR_+$, $W = \sum_{i=0}^{\infty} G_+^i K_+^{-1} R_+^i$.

Proof. The expressions for the matrix coefficients \tilde{A}_i , $i = -1, 0, 1$, follow from Theorem 2.4. Observe that, since $|\lambda| < 1$, from the definition of canonical factorization, it follows that the matrix $I - \lambda R_+$ is not singular. Therefore, since $A(\lambda)u = 0$ and also K_+ is nonsingular, from the canonical factorization of $A(z)$, we have $(I - \lambda^{-1}G_+)u = 0$. The existence and the expression of the canonical factorization of $\tilde{A}(z)$ can be obtained from Theorem 4.2 by setting $U(z) = (I - zR_+)K_+$ and $L(z) = I - zG_+$. In fact, Theorem 4.2 implies that $\tilde{A}(z)$ has the canonical factorization $\tilde{A}(z) = \tilde{U}(z)\tilde{L}(z)$ with $\tilde{U}(z) = U(z)$ and $\tilde{L}(z) = \tilde{L}_0 + z\tilde{L}_1$, where $\tilde{L}_0 = L_0 = I$ and $\tilde{L}_1 = L_1 - (\lambda - \mu)\lambda^{-1}L_1Q = -G_+ + (\lambda - \mu)\lambda^{-1}G_+Q$. Since $\lambda^{-1}G_+Q = Q$, we find that $\tilde{L}(z) = I - z(G_+ + (\mu - \lambda)Q)$. To prove

that $\tilde{A}(z^{-1})$ has a canonical factorization, we apply Theorem 4.1 to $\tilde{A}(z)$. To this purpose, we show that the matrix $\tilde{W} = \sum_{i=0}^{+\infty} \tilde{G}_+^i \tilde{K}_+^{-1} \tilde{R}_+^i$ is nonsingular. Observe that $\tilde{G}_+^i = G_+^i + (\mu - \lambda)Q G_+^{i-1}$, for $i \geq 1$. Therefore, since $\tilde{R}_+ = R_+$ and $\tilde{K}_+ = K_+$, we may write

$$\begin{aligned} \tilde{W} &= K_+^{-1} + \sum_{i=1}^{+\infty} (G_+^i + (\mu - \lambda)Q G_+^{i-1}) K_+^{-1} R_+^i \\ &= K_+^{-1} + \sum_{i=1}^{+\infty} G_+^i K_+^{-1} R_+^i + (\mu - \lambda)Q \left(\sum_{i=0}^{+\infty} G_+^i K_+^{-1} R_+^i \right) R_+ \\ &= W - (\lambda - \mu)Q W R_+. \end{aligned}$$

By Theorem 3.22 of [3] applied to $A(z)$, we have $\det W \neq 0$. Therefore $\det \tilde{W} = \det(I - (\lambda - \mu)Q W R_+ W^{-1}) \det W$. Moreover, since $Q = uv^*$, then the matrix $I - (\lambda - \mu)Q W R_+ W^{-1}$ is nonsingular if and only if

$$(\lambda - \mu)v^* W R_+ W^{-1} u \neq 1. \quad (4.3)$$

Since $G_- = W R_+ W^{-1}$, for Theorem 4.1 where $W = H_0$, condition (4.3) holds if $(\lambda - \mu)v^* G_- u \neq 1$, which is satisfied by assumption. Therefore the matrix \tilde{W} is nonsingular, so that from Theorem 4.1 applied to the matrix Laurent polynomial $\tilde{A}(z)$, we deduce that $\tilde{A}(z^{-1})$ has the canonical factorization $\tilde{A}(z^{-1}) = (I - z\tilde{R}_-)\tilde{K}_-(I - z^{-1}\tilde{G}_-)$ with $\tilde{K}_- = \tilde{A}_0 + \tilde{A}_{-1}\tilde{G}_- = \tilde{A}_0 + \tilde{R}_-A_1$ and $\tilde{G}_- = \tilde{W}R_+\tilde{W}^{-1}$, $\tilde{R}_- = \tilde{W}^{-1}\tilde{G}_+\tilde{W}$. \square

5 Some specific cases and applications

5.1 Left, right, and double shift

Observe that the Brauer theorem and its extensions can be applied to the matrix A^T or to the matrix polynomial $A(z)^T$. In this case we need to know a left eigenvector or a left invariant subspace of $A(z)$. The result is that we may shift a group of eigenvalues relying on the knowledge of a left invariant subspace. We refer to this procedure as to *left shift*. It is interesting to point out that right shift and left shift can be combined together. This combination is particularly convenient when dealing with palindromic polynomials. In fact, by shifting a group of eigenvalues by means of the right shift and the reciprocals of these eigenvalues by means of a left shift, we may preserve the palindromic structure.

Performing the double shift is also necessary when we are given a canonical factorization of $A(z)$ and we have to shift both an eigenvalue of modulus greater than 1 and an eigenvalue of modulus less than 1. This situation is encountered in the analysis of certain stochastic processes.

We describe the left shift as follows.

Theorem 5.1. Let $A(z) = \sum_{i \in \mathbb{Z}} z^i A_i$ be a matrix Laurent series analytic in the annulus \mathbb{A}_{r_1, r_2} . Let $\lambda \in \mathbb{A}_{r_1, r_2}$, be an eigenvalue of $A(z)$ such that $v^* A(\lambda) = 0$ for a nonzero vector $v \in \mathbb{C}^n$. Given $\mu \in \mathbb{C}$ define the matrix function

$$\tilde{A}(z) = \left(I + \frac{\lambda - \mu}{z - \lambda} S \right) A(z), \quad S = yv^*$$

where $y \in \mathbb{C}^n$ is such that $v^* y = 1$. Then $\tilde{A}(z)$ admits the power series expansion $\tilde{A}(z) = \sum_{i \in \mathbb{Z}} z^i \tilde{A}_i$, convergent for $z \in \mathbb{A}_{r_1, r_2}$, where

$$\begin{aligned} \tilde{A}_i &= A_i + (\lambda - \mu) \sum_{j=0}^{\infty} \lambda^j S A_{i+j+1}, \quad \text{for } i \geq 0, \\ \tilde{A}_i &= A_i - (\lambda - \mu) \sum_{j=0}^{\infty} \lambda^{-j-1} S A_{i-j}, \quad \text{for } i < 0. \end{aligned}$$

Moreover, we have $\det \tilde{A}(z) = \det A(z) \frac{z - \mu}{z - \lambda}$.

Proof. It is sufficient to apply Theorem 2.4 to the matrix function $A(z)^T$. \square

We provide a brief description of the double shift where two eigenvalues λ_1 and λ_2 are shifted by means of a right and a left shift respectively. We restrict our attention to the case where $\lambda_1 \neq \lambda_2$. The general case requires a more accurate analysis and is not treated here.

Theorem 5.2. Let $A(z) = \sum_{i \in \mathbb{Z}} z^i A_i$ be a matrix Laurent series analytic in the annulus \mathbb{A}_{r_1, r_2} . Let $\lambda_1, \lambda_2 \in \mathbb{A}_{r_1, r_2}$, $\lambda_1 \neq \lambda_2$, be two eigenvalues of $A(z)$ such that $A(\lambda_1)u = 0$, $v^* A(\lambda_2) = 0$ for nonzero vectors $u, v \in \mathbb{C}^n$. Given $\mu_1, \mu_2 \in \mathbb{C}$ define the matrix function

$$\tilde{A}(z) = \left(I + \frac{\lambda_2 - \mu_2}{z - \lambda_2} S \right) A(z) \left(I + \frac{\lambda_1 - \mu_1}{z - \lambda_1} Q \right), \quad Q = uw^*, \quad S = yv^* \quad (5.1)$$

where $w, y \in \mathbb{C}^n$ are such that $w^* u = 1$, $v^* y = 1$. Then $\tilde{A}(z)$ admits the power series expansion $\tilde{A}(z) = \sum_{i \in \mathbb{Z}} z^i \tilde{A}_i$, convergent for $z \in \mathbb{A}_{r_1, r_2}$, where

$$\begin{aligned} \tilde{A}_i &= \hat{A}_i + (\lambda_2 - \mu_2) \sum_{j=0}^{\infty} \lambda_2^j S \hat{A}_{i+j+1}, \quad \text{for } i \geq 0, \\ \tilde{A}_i &= \hat{A}_i - (\lambda_2 - \mu_2) \sum_{j=0}^{\infty} \lambda_2^{-j-1} S \hat{A}_{i-j}, \quad \text{for } i < 0, \end{aligned} \quad (5.2)$$

with

$$\begin{aligned} \hat{A}_j &= A_j + (\lambda_1 - \mu_1) \sum_{k=0}^{\infty} \lambda_1^k A_{k+j+1} Q, \quad \text{for } j \geq 0, \\ \hat{A}_j &= A_j - (\lambda_1 - \mu_1) \sum_{k=0}^{\infty} \lambda_1^{-k-1} A_{-k+j} Q, \quad \text{for } j < 0. \end{aligned} \quad (5.3)$$

Moreover, we have $\det \tilde{A}(z) = \det A(z) \frac{z-\mu_1}{z-\lambda_1} \frac{z-\mu_2}{z-\lambda_2}$.

Proof. Denote $\hat{A}(z) = A(z)(I + \frac{\lambda_1-\mu_1}{z-\lambda_1}Q)$, $Q = uw^*$. In view of Theorem 2.4, the matrix function $\hat{A}(z)$ is analytic in \mathbb{A}_{r_1, r_2} , moreover $\hat{A}(z) = \sum_{i \in \mathbb{Z}} z^i \hat{A}_i$ where \hat{A}_i are defined in (5.3). We observe that $v^* \hat{A}(\lambda_2) = v^* A(\lambda_2)(I + \frac{\lambda_1-\mu_1}{\lambda_2-\lambda_1}Q) = 0$ since $v^* A(\lambda_2) = 0$ and $\lambda_1 \neq \lambda_2$. Thus, we may apply Theorem 5.1 to the matrix function $\hat{A}(z)$ to shift the eigenvalue λ_2 to μ_2 and get the shifted function $\tilde{A}(z) = \sum_{i \in \mathbb{Z}} z^i \tilde{A}_i$ where the coefficients \tilde{A}_i are given in (5.2). The expression for the determinant follows by computing the determinant of both sides of (5.1). \square

In the above theorem, one can prove that applying the left shift followed by the right shift provides the same matrix Laurent series.

Observe that, if $A(z)$ is a matrix Laurent polynomial, i.e., $A(z) = \sum_{i=-h}^k z^i A_i$ then $\tilde{A}_i = 0$ for $i > k$ and $i < -h$, i.e., also $\tilde{A}(z)$ is a matrix Laurent polynomial. In particular, if $A(z) = z^{-1}A_{-1} + A_0 + zA_1$, then $\tilde{A}(z) = \sum_{i=-1}^1 z^i \tilde{A}_i$ with

$$\begin{aligned} \tilde{A}_1 &= A_1, \\ \tilde{A}_0 &= A_0 + (\lambda_1 - \mu_1)A_1Q + (\lambda_2 - \mu_2)SA_1, \\ \tilde{A}_{-1} &= A_{-1} - (\lambda_1 - \mu_1)\lambda_1^{-1}A_{-1}Q - (\lambda_2 - \mu_2)\lambda_2^{-1}S(A_{-1} - (\lambda_1 - \mu_1)\lambda_1^{-1}A_{-1}Q). \end{aligned}$$

If the matrix function admits a canonical factorization $A(z) = U(z)L(z^{-1})$ then we can prove that also the double shifted function $\tilde{A}(z)$ admits a canonical factorization. This result is synthesized in the next theorem.

Theorem 5.3. *Under the assumptions of Theorem 5.2, if $A(z)$ admits a canonical factorization $A(z) = U(z)L(z^{-1})$, with $U(z) = \sum_{i=0}^{\infty} z^i U_i$, $L(z) = \sum_{i=0}^{\infty} z^i L_i$, and if $|\lambda_1|, |\mu_1| < 1$, $|\lambda_2|, |\mu_2| > 1$, then $\tilde{A}(z)$ has the canonical factorization $\tilde{A}(z) = \tilde{U}(z)\tilde{L}(z^{-1})$, with $\tilde{U}(z) = \sum_{i=0}^{\infty} z^i \tilde{U}_i$, $\tilde{L}(z) = \sum_{i=0}^{\infty} z^i \tilde{L}_i$, where*

$$\tilde{L}_0 = L_0, \quad \tilde{L}_i = L_i - (\lambda_1 - \mu_1) \sum_{j=1}^{\infty} \lambda_1^{-j} L_{j+i-1} Q \quad i \geq 1,$$

$$\tilde{U}_i = U_i + (\lambda_2 - \mu_2) \sum_{j=0}^{\infty} \lambda_2^j S U_{i+j+1}, \quad i \geq 0.$$

Proof. Replacing $A(z)$ with $U(z)L(z^{-1})$ in equation (5.1) yields the factorization $\tilde{A}(z) = \tilde{U}(z)\tilde{L}(z^{-1})$ where $\tilde{U}(z) = (I + \frac{\lambda_2-\mu_2}{z-\lambda_2}S)U(z)$, and $\tilde{L}(z^{-1}) = L(z^{-1})(I + \frac{\lambda_1-\mu_1}{z-\lambda_1}Q)$. Since $v^*U(\lambda_2) = 0$ and $L(\lambda_1^{-1})u = 0$, we have that $\tilde{U}(z)$ and $\tilde{L}(z)$ are obtained by applying the left shift and the right shift to $U(z)$ and $L(z^{-1})$ respectively. Thus the expressions for $\tilde{U}(z)$ and $\tilde{L}(z)$ follow from Theorems 5.1 and 2.4, respectively. \square

5.2 Shifting eigenvalues from/to infinity

Recall that for a matrix polynomial having a singular leading coefficient, some eigenvalues are at infinity. These eigenvalues may cause numerical difficulties in certain algorithms for the polynomial eigenvalue problem. In other situations, having eigenvalues far from the unit circle, possibly at infinity, may increase the convergence speed of certain iterative algorithms. With the shift technique it is possible to move eigenvalues from/to infinity in order to overcome the difficulties encountered in solving specific problems. In this section we examine this possibility.

We restrict our attention to the case of matrix polynomials. Assume we are given $A(z) = \sum_{i=0}^d z^i A_i$ where A_d is singular, and a vector $u \neq 0$ such that $A_d u = 0$. Consider the reverse polynomial $A_{\mathcal{R}}(z) = \sum_{i=0}^d z^i A_{d-i}$, observe that $A_{\mathcal{R}}(0)u = A_d u = 0$ and that the eigenvalues of $A_{\mathcal{R}}(z)$ are the reciprocals of the eigenvalues of $A(z)$, with the convention that $1/0 = \infty$ and $1/\infty = 0$. Apply to $A_{\mathcal{R}}(z)$ a right shift of $\lambda = 0$ to a value $\sigma \neq 0$, and obtain a shifted polynomial $\tilde{A}_{\mathcal{R}}(z) = A_{\mathcal{R}}(z)(I - z^{-1}\sigma Q)$, with $Q = uv^*$, $v^*u = 1$, having the same eigenvalues of $A_{\mathcal{R}}(z)$ except for the null eigenvalue which is replaced by σ . Revert again the coefficients of the polynomial $\tilde{A}_{\mathcal{R}}(z)$ and obtain a new matrix polynomial $\tilde{A}(z) = \sum_{i=0}^d z^i \tilde{A}_i = z^d A_{\mathcal{R}}(z^{-1}) = A(z)(I - z\sigma Q)$. This polynomial has the same eigenvalues of $A(z)$ except for the eigenvalue at infinity which is replaced by $\mu = \sigma^{-1}$. A direct analysis shows that the coefficients of $\tilde{A}(z)$ are given by

$$\begin{aligned}\tilde{A}_0 &= A_0, \\ \tilde{A}_i &= A_i - \mu^{-1} A_{i-1} Q, \quad i = 1, \dots, d.\end{aligned}\tag{5.4}$$

A similar formula can be obtained for the left shift.

If the kernel of A_d has dimension $k \geq 1$ and we know a basis of the kernel, then we may apply the above shifting technique k times in order to remove k eigenvalues at infinity. This transformation does not guarantee that the leading coefficient of $\tilde{A}(z)$ is nonsingular. However, if $\det \tilde{A}_d = 0$ we may repeat the shift.

An example of this transformation is given by the following quadratic matrix polynomial

$$A(z) = z^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + z \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

which has two eigenvalues at infinity, and 4 eigenvalues equal to $\pm i\sqrt{3}$, $\pm i$, where i is the imaginary unit. The Matlab function `polyeig` applied to this polynomial yields the following approximations to the eigenvalues

$$\begin{aligned}&\text{Inf} + 0.0000000000000000\text{e}+00i \\ &1.450878662283769\text{e}-15 + 1.732050807568876\text{e}+00i\end{aligned}$$

$$\begin{aligned}
& 1.450878662283769\text{e-15} - 1.732050807568876\text{e+00i} \\
& -8.679677645252271\text{e-17} + 9.999999999999998\text{e-01i} \\
& -8.679677645252271\text{e-17} - 9.999999999999998\text{e-01i} \\
& \text{Inf} + 0.000000000000000\text{e+00i}.
\end{aligned}$$

Since $e_1 = (1, 0, 0)^*$ is such that $A_2 e_1 = 0$, we may apply (5.4) with $\mu = 1$ and $Q = e_1 e_1^*$ yielding the transformed matrix polynomial

$$z^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + z \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

which still has the same eigenvalues of $A(z)$ except one eigenvalue at infinity which is replaced by 1. Since e_1 is still in the kernel of the leading coefficient, we may perform again the shift to the above matrix polynomial with $Q = e_1 e_1^*$ and $\mu = 1/2$ yielding the matrix polynomial

$$z^2 \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} + z \begin{bmatrix} -3 & 1 & 1 \\ -3 & 1 & 1 \\ -3 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

which has no eigenvalues at infinity. Applying again `polyeig` to this latter polynomial yields

$$\begin{aligned}
& 5.000000000000000\text{e-01} + 0.000000000000000\text{e+00i} \\
& 9.999999999999993\text{e-01} + 0.000000000000000\text{e+00i} \\
& -1.456365117861229\text{e-16} + 1.732050807568876\text{e+00i} \\
& -1.456365117861229\text{e-16} - 1.732050807568876\text{e+00i} \\
& -4.885759970651636\text{e-17} + 9.999999999999996\text{e-01i} \\
& -4.885759970651636\text{e-17} - 9.999999999999996\text{e-01i}.
\end{aligned}$$

It is evident that eigenvalues $\pm i$ and $\pm i\sqrt{3}$ are left unchanged, while the two eigenvalues of the original polynomial at infinity are replaced by 1 and by $1/2$.

A similar technique can be applied to shift a finite eigenvalue of $A(z)$ to infinity. Assume that $A(\lambda)u = 0$ for $u \neq 0$ where $\lambda \neq 0$. Consider the reversed polynomial $A_{\mathcal{R}}(z) = z^d A(z^{-1})$ and apply the right shift to $A_{\mathcal{R}}(z)$ to move λ^{-1} to 0. This way we obtain the polynomial $\tilde{A}_{\mathcal{R}}(z) = A_{\mathcal{R}}(z)(I + \frac{\lambda^{-1}}{z-\lambda^{-1}}Q)$, $Q = uv^*$, with v any vector such that $v^*u = 1$. Revert again the coefficients of the polynomial $\tilde{A}_{\mathcal{R}}(z)$ and obtain a new matrix polynomial $\tilde{A}(z) = \sum_{i=0}^d z^i \tilde{A}_i = z^d A_{\mathcal{R}}(z^{-1}) = A(z)(I + \frac{\lambda^{-1}}{z-1-\lambda^{-1}}Q)$. The equations which relate the coefficients of $A(z)$ and $\tilde{A}(z)$ can be obtained by reverting the coefficients of $\tilde{A}_{\mathcal{R}}(z)$ and relying on Theorem 2.2. This way we get

$$\begin{aligned}
\tilde{A}_0 &= A_0, \\
\tilde{A}_i &= A_i + \lambda^{-1} \sum_{k=0}^{i-1} \lambda^{-k} A_{i-k+1} Q, \quad i = 1, \dots, d.
\end{aligned} \tag{5.5}$$

A similar expression can be obtained with the left shift. In the transformed matrix polynomial, the eigenvalue λ is shifted to infinity.

5.3 Palindromic polynomials

Consider a $*$ -palindromic matrix polynomial $A(z) = \sum_{i=0}^d z^i A_i$, that is, such that $A_i = A_{d-i}^*$ for $i = 0, \dots, d$, and observe that $A(z)^* = \bar{z}^d A(\bar{z}^{-1})$. This way, one has $A(\lambda)u = 0$ for some $\lambda \in \mathbb{C}$ and some vector $u \neq 0$, if and only if $u^* A(\bar{\lambda}^{-1}) = 0$. Thus the eigenvalues of $A(z)$ come into pairs $(\lambda, \bar{\lambda}^{-1})$, where we assume that $1/0 = \infty$ and $1/\infty = 0$.

If we are given λ and u such that $A(\lambda)u = 0$, we can apply the right shift to move λ to some value μ and the left shift to the polynomial $A(z)^*$, thus shifting $\bar{\lambda}$ to $\bar{\mu}$. For $\mu \in \mathbb{C}$, the functional expression of this double shift is

$$\tilde{A}(z) = \left(I + \frac{\bar{\lambda} - \bar{\mu}}{z^{-1} - \bar{\lambda}} Q \right) A(z) \left(I + \frac{\lambda - \mu}{z - \lambda} Q \right),$$

where $Q = uu^*/(u^*u)$. We may easily check that $\tilde{A}(z)^* = \bar{z}^d \tilde{A}(\bar{z}^{-1})$, i.e., $\tilde{A}(z)$ is $*$ -palindromic. Moreover, one has

$$\det \tilde{A}(z) = \left(1 + \frac{\bar{\lambda} - \bar{\mu}}{z^{-1} - \bar{\lambda}} \right) \det A(z) \left(1 + \frac{\lambda - \mu}{z - \lambda} \right) = \det A(z) \frac{(z - \mu)(1 - \bar{\mu}z)}{(z - \lambda)(1 - \bar{\lambda}z)},$$

hence the pair of eigenvalues $(\lambda, \bar{\lambda}^{-1})$ is moved to the pair $(\mu, \bar{\mu}^{-1})$. The coefficients of the shifted matrix polynomial $\tilde{A}(z) = \sum_{i=0}^d \tilde{z}^i A_i$ are given by

$$\tilde{A}_0 = \hat{A}_0, \quad \tilde{A}_i = \hat{A}_i + (\bar{\lambda} - \bar{\mu}) \sum_{k=0}^{i-1} \bar{\lambda}^k Q \hat{A}_{i-k-1}, \quad i = 1, \dots, d,$$

where

$$\hat{A}_d = A_d, \quad \hat{A}_j = A_j + (\lambda - \mu) \sum_{k=0}^{d-j-1} \lambda^k A_{k+j+1} Q, \quad j = 0, \dots, d-1.$$

In the particular case where $\lambda = 0$, the expression for the coefficients simplifies to

$$\tilde{A}_i = A_i - \mu A_{i+1} Q - \bar{\mu} Q A_{i-1} + |\mu|^2 Q A_i Q, \quad i = 0, \dots, d,$$

with the convention that $A_{-1} = A_{d+1} = 0$.

In the specific case where $A(z)$ is a matrix polynomial of degree 2, the coefficients of $\tilde{A}(z)$ have the simpler expression:

$$\begin{aligned} \tilde{A}_0 &= A_0 + (\lambda - \mu)(A_1 + \lambda A_2)Q, \\ \tilde{A}_1 &= A_1 + (\lambda - \mu)A_2 Q + (\bar{\lambda} - \bar{\mu})Q A_0 + |\lambda - \mu|^2 Q(A_1 + \lambda A_2)Q, \\ \tilde{A}_2 &= \tilde{A}_0^*. \end{aligned}$$

In the above formulas, since $(A_1 + \lambda A_2)Q = -\lambda^{-1}A_0 Q$ and $Q(A_1 + \bar{\lambda}A_0) = -\bar{\lambda}^{-1}Q A_2$, we may easily check that $\tilde{A}_1^* = \tilde{A}_1$.

Concerning canonical factorizations, we may prove that if $d = 2m$ and

$$z^{-m}A(z) = U(z)L(z^{-1}), \quad U(z) = L(\bar{z})^*,$$

is a canonical factorization of $z^{-m}A(z)$, and if $|\lambda| < 1, |\mu| < 1$, then the shifted function $z^{-m}\tilde{A}(z)$ has the canonical factorization

$$z^{-m}\tilde{A}(z) = \tilde{U}(z)\tilde{L}(z^{-1}), \quad \tilde{U}(z) = \tilde{L}(\bar{z})^*, \quad \tilde{L}(z^{-1}) = L(z^{-1})(I + \frac{\lambda - \mu}{z - \lambda}Q).$$

5.4 Quadratic matrix polynomials and matrix equations

Recall that [3] a matrix Laurent polynomial of the kind $A(z) = z^{-1}A_{-1} + A_0 + zA_1$ has a canonical factorization

$$A(z) = (I - zR_+)K_+(I - z^{-1}G_+)$$

if and only if the matrices R_+ and G_+ are the solutions of the matrix equations

$$X^2A_{-1} + XA_0 + A_1 = 0, \quad A_{-1} + A_0X + A_1X^2 = 0,$$

respectively, with spectral radius less than 1. From Theorem 4.3 it follows that the matrices $\tilde{R}_+ = R_+$ and $\tilde{G}_+ = G_+ + (\lambda - \mu)Q$ are the solutions of spectral radius less than 1 to the following matrix equations

$$X^2\tilde{A}_{-1} + X\tilde{A}_0 + \tilde{A}_1 = 0, \quad \tilde{A}_{-1} + \tilde{A}_0X + \tilde{A}_1X^2 = 0,$$

respectively, where $\tilde{A}(z) = A(z)(I + \frac{\lambda - \mu}{z - \lambda}Q)$ is the matrix Laurent polynomial obtained with the right shift with $|\lambda|, |\mu| < 1$.

If in addition $A(z^{-1})$ has the canonical factorization

$$A(z^{-1}) = (I - zR_-)K_-(I - z^{-1}G_-),$$

and the assumptions of Theorem 4.3 are satisfied, then the matrices \tilde{R}_- and \tilde{G}_- defined in Theorem 4.3 are the solution of spectral radius less than 1 to the matrix equations

$$X^2\tilde{A}_1 + X\tilde{A}_0 + \tilde{A}_{-1} = 0, \quad \tilde{A}_1 + \tilde{A}_0X + \tilde{A}_{-1}X^2 = 0,$$

respectively.

For a general matrix polynomial $A(z) = \sum_{i=0}^d z^i A_i$ the existence of the canonical factorization $z^{-1}A(z) = U(z)(I - z^{-1}G)$ implies that G is the solution of minimal spectral radius of the equation

$$\sum_{i=0}^d A_i X^i = 0.$$

Let λ be an eigenvalue of $A(z)$ of modulus less than 1 such that $A(\lambda)u = 0$, $u \in \mathbb{C}^n$, $u \neq 0$ and let $\mu \in \mathbb{C}$ be such that $|\mu| < 1$. Consider the polynomial

$\tilde{A}(z)$ defined in Theorem 2.2 obtained by means of the right shift. According to Theorem 4.2 the function $z^{-1}\tilde{A}(z)$ has the canonical factorization $z^{-1}\tilde{A}(z) = U(z)(I - z^{-1}\tilde{G})$ where $\tilde{G} = G - (\lambda - \mu)Q$. Therefore the matrix \tilde{G} is the solution of minimal spectral radius of the matrix equation

$$\sum_{i=0}^d \tilde{A}_i X^i = 0.$$

As an example of application of this result, consider the $n \times n$ matrix polynomial $A(z) = zI - B(z)$ where $B(z) = \sum_{i=0}^d z^i B_i$ with nonnegative coefficients such that $(\sum_{i=0}^d B_i)e = e$, $e = (1, \dots, 1)^*$. Choose $n = 5$, $d = 4$ and construct the coefficients B_i as follows. Let $E = (e_{i,j})$, $e_{i,j} = 1$, $T = (t_{i,j})$ be the upper triangular matrix such that $t_{i,j} = 1$ for $i \leq j$, $D = \text{diag}(57, 49, 41, 33, 25)$, set $B_0 = 9D^{-1}T$, $B_1 = D^{-1}T^*$, $B_2 = D^{-1}E$, $B_3 = D^{-1}E$, $B_4 = D^{-1}$. Then $B(1)e = e$ so that $A(1)e = 0$. The matrix polynomial $A(z)$ has 5 eigenvalues of modulus less than or equal to 1, that is

$$\begin{aligned} &0.15140 - 0.01978i \\ &0.15140 + 0.01978i \\ &0.20936 - 0.09002i \\ &0.20936 + 0.09002i \\ &1.00000 + 0.00000i \end{aligned}$$

moreover, the eigenvalue of smallest modulus among those which lie outside the unit disk is 1.01258. Applying cyclic reduction to approximate the minimal solution of the matrix equation $\sum_{i=0}^4 A_i X^i = 0$, relying on the software of [6], requires 12 iterations. Applying the same method with the same software to the equation $\sum_{i=0}^4 \tilde{A}_i X^i = 0$ obtained by shifting the eigenvalue 1 to zero provides the solution in just 6 iterations. In fact the approximation error at step k goes to zero as σ^{2^k} where σ is the ratio between the eigenvalue of largest modulus, among those in the unit disk, and the eigenvalue of smallest modulus, among those out of the unit disk. This ratio is 0.98758 for the original polynomial and 0.20676 for the polynomial obtained after the shift. This explains the difference in the number of iterations.

6 Conclusions and open problems

By means of a functional interpretation, we have extended Brauer's theorem to matrix (Laurent) polynomials, matrix (Laurent) power series and have related the canonical factorizations of these matrix functions to the corresponding factorizations of the shifted functions.

One point that has not been treated in this article is the analysis of the Jordan chains of a matrix polynomial or matrix power series under the right/left or double shift. This is the subject of next investigation.

Another interesting issue which deserves further study is the formal analysis of the conditioning of the eigenvalues of a matrix polynomial under the shift,

together with the analysis of the conditioning of the shift operation itself and of the numerical stability of the algorithms for its implementation.

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